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Construction of Continuous-Time Equivalents of Welch Bound Equality Sequences

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Abstract

Recently, the Welch bound equality (WBE) sequences and their variations have been intensively investigated for applications to code-division multiple-access (CDMA) communications and, accordingly, many algorithms to construct these sequences have been proposed. In this paper, we propose an algorithm called multi-user piecewise-constant constrained water-filling to construct the continuous-time equivalents of Welch bound equality (CTE-WBE) sequences, which are continuous-time counterparts of the WBE sequences.

1. INTRODUCTION

The Welch bound equality (WBE) sequences [1] are defined as K unit-energy vectors $(\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_K)$ of length $N(\leq K)$, which jointly achieve the lower bound on the total squared correlation (TSC) given by

$$\sum_{k=1}^{K}\sum_{k'=1}^{K} |\mathbf{s}_k^T \mathbf{s}_{k'}|^2 \geq \frac{K^2}{N} \tag{1}$$

where the superscript ^{T} denotes transposition. It is well known [2] that the members in a set of vectors are WBE sequences if and only if the condition

$$\mathbf{S}\mathbf{S}^T = \frac{K}{N}\mathbf{I}_N \tag{2}$$

is met by $\mathbf{S} \triangleq [\mathbf{s}_1\ \ \mathbf{s}_2\ \ ...\ \ \mathbf{s}_K]$, where $\mathbf{I}_N$ is the N-dimensional identity matrix.

Recently, these WBE sequences and their variations such as the generalized WBE (GWBE) sequences have found many applications to code-division multiple-access (CDMA) communications when used as signature sequences: In [2], WBE sequences are shown to minimize the total signal power subject to equal signal-to-interference-plus-noise ratio (SINR) constraints at the output of linear minimum mean-squared error (LMMSE) receivers; In [3], WBE sequences are shown to maximize the sum capacity of equal-power users; The GWBE sequences are shown in [2] to minimize the total signal power subject to unequal SINR constraints at the output of LMMSE receivers and in [4] to maximize the sum capacity for unequal-power users. Wavelet applications also have been found in [2] and [5], respectively, by showing that the WBE sequences and the GWBE sequences form tight frames [6] for the finite-dimensional Euclidean spaces. Accordingly, various algorithms also have been developed [7]–[11] to construct these sequence sets.

In this paper, we provide an algorithm to construct a set of waveforms called the continuous-time equivalents of Welch bound equality (CTE-WBE) sequences [12]. Similar to the WBE sequences, the CTE-WBE sequences minimize the continuous-time equivalent of total squared correlation (CTE-TSC) and minimize the total mean-squared error (TMSE) at the output of the linear receivers for *continuous-time* CDMA systems having equal-power users. It is shown that the symbol waveforms for the optimal time-division multiple-access (TDMA) systems and those for the optimal frequency-division multiple-access (FDMA) systems that, respectively, minimize the TMSE at the linear receivers are also CTE-WBE sequences. Although the work in [12] fully characterized a necessary and sufficient condition for a set of waveforms to be a set of CTE-WBE sequences and provided some trivial example sets of CTE-WBE sequences, the question on how to construct non-trivial sets of CTE-WBE sequences has been remained open. The multi-user piecewise-constant constrained water-filling algorithm proposed in this paper is the first algorithm to answer this question.

The organization of this paper is as follows. In Section II, we briefly review the definition of the CTE-WBE sequences and formulate the algorithm design problem. In Section III, we propose an algorithm called multi-user piecewise-constant constrained water-filling

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that, together with sets of GWBE sequences, leads to the CTE-WBE sequences. The concluding remarks are offered in Section IV.

2. Review of Definitions and Problem Formulation

The CTE-WBE sequences are defined as K strictly band-limited, unit-energy waveforms $(s_1(t), s_2(t), ..., s_K(t))$ of bandwidth $W/2$ [Hz] in baseband, which jointly achieve the lower bound derived in [12] on the continuous-time equivalent of total squared correlation (CTE-TSC) given by

$$\sum_{k=1}^{K}\sum_{k'=1}^{K}\left(\sum_{l=-\infty}^{\infty}\left|s_k(-t)^* * s_{k'}(t)|_{l=lT}\right|^2\right) \geq \frac{K^2}{WT} \tag{3}$$

where the superscript * denotes complex conjugation, the operator $*$ denotes convolution, and $T(< K/W)$ is the symbol-time when these waveforms are used as the signature waveforms of multiple-access systems.

To fully characterize a set of CTE-WBE sequences, we need the following definition.

Definition 1 *[13,* Definition 5*] The vectorized Fourier transform* $\mathbf{x}(f)$ *of the time function* $x(t)$ *with sampling rate* T *[samples/Hz] is given by*

$$\mathbf{x}(f) = \begin{bmatrix} X\left(-\frac{L}{T}+f\right) \\ X\left(-\frac{L-1}{T}+f\right) \\ \cdots \\ X\left(\frac{L}{T}+f\right) \end{bmatrix} \tag{4}$$

for $f \in [-1/(2T), 1/(2T))$*, where* $X(\xi)$ *is the Fourier transform of* $x(t)$ *defined as* $X(\xi) \triangleq \int_{-\infty}^{\infty} x(t)e^{-\mathrm{j}2\pi\xi t}dt$*, and* $L \triangleq \lceil \beta/2 \rceil = \lceil (WT-1)/2 \rceil$ *is determined by the excess bandwidth, equivalently, by the bandwidth* W *and the symbol rate* $1/T$*.*

Since not all of the entries in $\mathbf{x}(f)$ are the decision variables due to the limited bandwidth of the signature waveforms, the following refinement on the definition of the vectorized Fourier transform is required.

Definition 2 *[13,* Definition 7*] The effective vectorized Fourier transform of* $x(t)$ *is defined as a variable-dimensional function of* f *that is obtained after removing the first entry of* $\mathbf{x}(f)$ *for* $-1/(2T) \leq f < -(1+\beta)/(2T) + L/T$ *and the last entry of* $\mathbf{x}(f)$ *for* $(1+\beta)/(2T) - L/T \leq f < 1/(2T)$*.*

In what follows, every vectorized Fourier transform is an effective one.

It is shown in [12] that the members in a set of strictly band-limited, unit-energy waveforms are CTE-WBE sequences if and only if the condition

$$\frac{1}{T}\mathbf{S}(f)\mathbf{S}(f)^H = \frac{K}{WT}\mathbf{I}_{N(f)} \tag{5}$$

is met for $f \in [-1/(2T), 1/(2T))$ by $\mathbf{S}(f) \triangleq [\mathbf{s}_1(f)\ \mathbf{s}_2(f)\ ...\ \mathbf{s}_K(f)]$, where $\mathbf{s}_k(f)$ is the vectorized Fourier transform of the waveform $s_k(t)$, $N(f)$ is the dimension of the vectorized Fourier transforms as a function of the offset f given by [12]

$$N(f) = \begin{cases} 1 + \lceil WT-1 \rceil, & \text{for } |f| < \frac{WT-\lceil WT-1 \rceil}{2T} \\ \lceil WT-1 \rceil, & \text{otherwise} \end{cases} \tag{6a}$$

for even $\lceil WT-1 \rceil$ and

$$N(f) = \begin{cases} \lceil WT-1 \rceil, & \text{for } |f| < \frac{\lceil WT-1 \rceil - WT-1}{2T} \\ 1 + \lceil WT-1 \rceil, & \text{otherwise} \end{cases} \tag{6b}$$

for odd $\lceil WT-1 \rceil$, and $\mathbf{I}_{N(f)}$ is the $N(f)$-dimensional identity matrix-valued function of $f \in [-1/(2T), 1/(2T))$.

By comparing (1) to (3), and (2) to (5), we can easily notice the striking resemblance between the WBE sequences and the CTE-WBE sequences. Actually, if $N = WT$ and each sequence $\mathbf{s}_k$ linearly modulates uniformly-delayed sinc pulses with bandwidth $W/2$ and unit delay T/N, then (3) reduces to (1), and (5) reduces to (2). So, the CTE-WBE sequences are the extensions of the WBE sequences to continuous-time as well as the continuous-time counterparts of the WBE sequences. Moreover, if $WT = K$, the condition (5) reduces to the generalized Nyquist condition [14, Eq. (6.56)] for zero ISI and zero mutual interference. Thus, the CTE-WBE sequences can be regarded as quasi-orthogonal signals that generalizes the Nyquist pulses.

Although, in [12], the CTE-WBE sequences were defined and shown to be the optimal signature waveforms that minimize the total mean-squared error at the output of linear receivers for Gaussian multiple-access communications, no algorithm was proposed to construct these waveforms, but just a few examples were provided. Therefore, it is of interest in this paper how to construct these waveforms using a simple algorithm. In the next section, we propose an algorithm called multi-user piecewise-constant constrained water-filling for this purpose.

3. Multi-User Piecewise-Constant Constrained Water-Filling

The construction of the CTE-WBE sequences are performed in the frequency domain. To proceed, we introduce the notions of the differential signal power and the normalized vectorized Fourier transforms of the waveforms $(s_k(t))_k$.

Definition 3 *Given the vectorized Fourier transform* $\mathbf{s}_k(f)$ *of* $s_k(t)$*, the differential signal power* $P_k(f)$ *and the normalized vectorized Fourier transform* $\mathbf{z}_k(f)$ *are defined, respectively, by*

$$P_k(f) \triangleq \frac{\|\mathbf{s}_k(f)\|^2}{T} \tag{7a}$$

and

$$\mathbf{z}_k(f) \triangleq \frac{\mathbf{s}_k(f)}{\|\mathbf{s}_k(f)\|^2} \tag{7b}$$

for $f \in [-1/(2T), 1/(2T))$.

Note that, as $\mathbf{s}_k(f)$ completely determines the pair $(P_k(f), \mathbf{z}_k(f))$ by (7a) and (7b), the pair completely determines $\mathbf{s}_k(f)$ as

$$\mathbf{s}_k(f) = \sqrt{P_k(f)T}\mathbf{z}_k(f). \tag{8}$$

Thus, instead of directly obtaining the set of the vectorized Fourier transforms $(\mathbf{s}_k(f))_{k=1}^K$, we divide the construction procedure into two steps, where we first construct a feasible profile of $(P_k(f))_{k=1}^K$ and then construct a profile $(\mathbf{z}_k(f))_{k=1}^K$ of normalized vectorized Fourier transforms.

- *Step 1:* Construction of a feasible profile $(P_k(f))_{k=1}^K$ of differential signal power

(S1a) Divide the interval $f \in \{f : N(f) = 1 + \lceil WT - 1 \rceil\}$ into $2M_1 (\geq 2)$ equal-length intervals and the interval $f \in \{f : N(f) = \lceil WT - 1 \rceil\}$ into $2M_2 (\geq 2)$ equal-length intervals.

(S1b) Let f_m be the mid-point of the mth sub-interval for $m = 1, 2, ..., 2M_1 + 2M_2$ and L_m be the length of the mth sub-interval for $m = 1, 2, ..., 2M_1 + 2M_2$.

(S1c) Set $k = 0$.

(S1d) $k := k + 1$.

(S1e) Choose any profile $(P_k(f_m))_{m=1}^{2M_1+2M_2}$ such that

$$0 \leq P_k(f_m) \leq \frac{K}{WT}, \forall m, \tag{9a}$$

$$\sum_{m=1}^{2M_1+2M_2} P_k(f_m)L_m = \frac{1}{T}, \tag{9b}$$

$$\text{and} \sum_{k'=1}^{k} P_{k'}(f_m) \leq \frac{KN(f_m)}{WT}, \forall m. \tag{9c}$$

(S1f) If $k < K$, then goto (S1d).

(S1g) In the mth interval, set $P_k(f) = P_k(f_m)$ for $k = 1, 2, ..., K$ and $m = 1, 2, ..., 2M_1 + 2M_2$.

- *Step 2:* Construction of a profile $(\mathbf{z}_k(f))_{k=1}^K$ of normalized vectorized Fourier transforms

(S2a) Set $m = 0$.

(S2b) $m := m + 1$.

(S2c) Assign GWBE sequences to $(\mathbf{z}_k(f_m))_{k=1}^K$ such that

$$\sum_{k=1}^{K} P_k(f_m)\mathbf{z}_k(f_m)\mathbf{z}_k(f_m)^H = \left(\frac{\sum_{k=1}^{K} P_k(f_m)}{N(f_m)}\right)\mathbf{I}_{N(f_m)} \tag{10}$$

(S2d) If $m < 2M_1 + 2M_2$, then goto (S2b).

(S2e) In the mth interval, set $\mathbf{z}_k(f) = \mathbf{z}_k(f_m)$ for $k = 1, 2, ..., K$ and $m = 1, 2, ..., 2M_1 + 2M_2$.

In what follows, we show that the above algorithm produce the waveforms in a set of the CTE-WBE sequences.

Lemma 1 *The total differential signal power of the proposed algorithm is given by*

$$\sum_{k=1}^{K} P_k(f_m) = \frac{KN(f_m)}{WT}, \forall m. \tag{11}$$

Proof: By (9b) and (9c), we have

$$\frac{K}{T} = \sum_{k=1}^{K}\left(\sum_{m=1}^{2N_1+2M_2} P_k(f_m)L_m\right) \tag{12a}$$

$$= \sum_{m=1}^{2N_1+2M_2}\left(\sum_{k=1}^{K} P_k(f_m)\right)L_m \tag{12b}$$

$$\leq \sum_{m=1}^{2N_1+2M_2}\left(\frac{KN(f_m)}{WT}\right)L_m \tag{12c}$$

$$= \frac{K}{WT}\sum_{m=1}^{2N_1+2M_2} N(f_m)L_m = \frac{K}{T} \tag{12d}$$

where the result $\int_{-1/(2T)}^{1/(2T)} N(f)df = W$ in [12] is used in (12d). Since the equality in (12c) holds if and only if (11) is true, the conclusion follows.

This lemma enables a geometric interpretation of *Step 1* in the proposed algorithm. Now, *Step 1* can be viewed as a successive pouring of different liquids to fill a basin having depth $N(f)$ for $f \in [-1/(2T), 1/(2T))$. The kth liquid, for $k = 1, 2, ..., K$, has the total volume of $1/T$, and its thickness $P_k(f)$ at any f must not exceed $K/(WT)$. Although the order of pouring at each f actually does not matter, the total depth $\sum_{k=1}^{K} P_k(f)$ of all the liquids at each f must be $KN(f)/(WT)$. Since f is a member of an uncountable set, we divide the interval of f into $2M_1 + 2M_2$ finite subintervals and make $P_k(f)$ piecewise constant to enable this procedure to end in a finite number of steps. This procedure is similar to the classical water-pouring that maximizes the channel capacity of a parallel AWGN channel [15], but it has additional constraints. Thus, we term the proposed algorithm as the multi-user piecewise-constant constrained water-filling.

Now, it suffices to show that *Step 2* indeed produce a set of waveforms that satisfy the condition (5).

Lemma 2 *Given* $(P_1, P_2, ..., P_K)$ *and* N *with* $P_k \geq 0, \forall k$ *and* $N \in \mathbb{N}$*, a set of sequences* $(\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_K)$ *with* $\mathbf{z}_k \in \mathbb{C}^N$*,* $\|\mathbf{z}_k\| = 1$*, and*

$$\sum_{k=1}^{K} P_k \mathbf{z}_k \mathbf{z}_k^H = \left(\frac{\sum_{k=1}^{K} P_k}{N} \right) \mathbf{I}_N \tag{13a}$$

exists if and only if

$$\frac{\sum_{k'=1}^{K} P_{k'}}{N} \geq P_k, \forall k, \text{ and } \sum_{k=1}^{K} sgn(P_k) \geq N \tag{13b}$$

Proof: See Appendix of [2].

Note that how to construct such a set of sequences that satisfy (13a) is given in [4].

Proposition 1 *The proposed algorithm produces the CTE-WBE sequences.*

Proof: By *Lemma 1* and (9a),

$$\frac{\sum_{k=1}^{K} P_k(f_m)}{N(f_m)} = \frac{K}{WT} \geq P_k(f_m), \forall k. \tag{14}$$

If $\sum_{k=1}^{K} \text{sgn}(P_K(f_m)) < N(f_m)$, then by (9a) $\sum_{k=1}^{K} P_K(f_m) < KN(f_m)/(WT)$, which contradicts *Lemma 1*. Thus, $\sum_{k=1}^{K} \text{sgn}(P_K(f_m))$ must be greater than or equal to $N(f_m), \forall m$. Therefore, the conditions in (13b) are all met for the existence of the GWBE sequences that satisfy the condition (10) in *Step 2* for each m. Now, by (8), the left-side of (5) becomes

$$\begin{aligned} \sum_{k=1}^{K} P_k(f_m) \mathbf{z}_k(f_m) \mathbf{z}_k(f_m)^H &= \frac{1}{T} \sum_{k=1}^{K} \mathbf{s}_k(f_m) \mathbf{s}_k(f_m)^H \\ &= \frac{1}{T} \mathbf{S}(f_m) \mathbf{S}(f_m)^H, \forall m \end{aligned} \tag{15}$$

Moreover, by *Lemma 1*, the right-side of (5) becomes

$$\left(\frac{\sum_{k=1}^{K} P_k(f_m)}{N(f_m)} \right) \mathbf{I}_{N(f_m)} = \frac{K}{WT} \mathbf{I}_{N(f_m)}, \forall m. \tag{16}$$

Thus, the condition (5) is met. Therefore, the conclusion follows.

4. CONCLUSIONS

In this paper, an algorithm called the multi-user piecewise-constant constrained water-filling is proposed to construct a set of waveforms that are the continuous-time counterparts of the celebrated WBE sequences. The algorithm optimally distributes the signal power using a geometric procedure similar to the classical water-filling and utilizes a finite number of sets of GWBE sequences in the construction of the waveforms.

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