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Asymptotic Equivalence of SC LMMSE-FDE to Continuous-Time LMMSE Equalizer

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***Abstract*— In this paper, it is shown that single-carrier (SC) linear minimum mean-squared error (LMMSE) frequency domain equalizer (FDE) is asymptotically equivalent to continuous-time (CT) LMMSE equalizer in the limit of a very large block length. The equivalence is established in the sense that the mean-squared error (MSE) of the SC LMMSE-FDE converges to that of the CT LMMSE equalizer. First, the asymptotic MSE of the SC LMMSE-FDE is derived. Then, the necessary and sufficient condition for the asymptotic equivalence is obtained. It is shown that if the received signal is sampled slower than the Nyquist rate then *no* fixed receive filter followed by a symbol-rate sampler can achieve the MSE of the CT LMMSE equalizer, given an arbitrary channel response. On the contrary, if the received signal is sampled faster than the Nyquist rate then the MSE of the CT LMMSE receiver can be asymptotically achieved by *any* square-root Nyquist receive filter, of which energy spectral density is flat over transmitted signal's frequency band. Representative numerical results are provided and discussions on convergence rate are also provided.**

I. Introduction

As the data rate in communication systems increases, the frequency selectivity in channel induces severe distortion in the received signal. The orthogonal frequency division multiplexing (OFDM) has been considered as one of the most promising modulation schemes for communication over frequency selective channels. However, the OFDM suffers from high peak-to-average power ratio (PAPR) and high sensitivity to carrier synchronization error. The single-carrier (SC) block transmission with cyclic prefix (CP) is another promising modulation scheme, which does not suffer from high PAPR but performs comparably to the OFDM receiver when equalized in the frequency domain [1], [2]. Among various optimality criteria for the frequency domain equalizer (FDE), we consider linear minimum mean-squared error (LMMSE) criterion in this paper.

In conventional SC linear modulation, data symbols are continuously transmitted. If the LMMSE criterion is applied to minimize the mean-squared error (MSE) between the original information symbols and their estimates, the continuous-time (CT) LMMSE equalizer is obtained that consists of a linear filter followed by a symbol-rate sampler. The linear filter has the frequency response that is dependent on the transmit pulse and the channel impulse response. To adapt to different channels, it is required in theory to control the receive filter in the frequency domain with infinite resolution. In practice, the receiver structure consisting of a fixed receive filter followed by a sampler that over-samples the filter output and an adaptive equalizer is commonly used. The well-known fractionally-spaced LMMSE equalizer has such a structure and it is shown that its MSE converges to that of the CT LMMSE equalizer as the tap length and, consequently, the observation interval tends to infinity [3].

In SC block transmission with CP, an SC FDE processes the received signal block by block, where the data samples corresponding to the CP of the first arriving path are discarded to make the channel matrix circulant. When the block length is large, the decision statistic is different from that of the fractionally-spaced LMMSE equalizer with a large tap length only in the samples at the beginning of each block. Thus, by resorting to the asymptotic equivalence of the fractionally-spaced LMMSE equalizer to the CT LMMSE equalizer, it has been claimed without a formal proof that the SC LMMSE-FDE also has the same asymptotic MSE performance as the CT LMMSE equalizer in the limit of a very large block length [4], [5].

In this paper, we prove that the SC LMMSE-FDE is asymptotically equivalent to the CT LMMSE equalizer under certain conditions, and identify these conditions on the receive filter and the over-sampling factor. The rest of this paper is organized as follows. In Section II, the system model for the SC LMMSE-FDE is described and its MSE is derived for various block lengths and over-sampling factors. In Section III, the necessary and sufficient condition for the asymptotic equivalence is obtained. The asymptotic equivalence is proved in the sense that the MSE of the SC LMMSE-FDE converges to that of the CT LMMSE equalizer as the block length tends to infinity. In Section IV, representative numerical results that support the theoretical analysis and discussions on the convergence rate are provided. Finally, concluding remarks are offered in Section V.

Throughout this paper, boldface lowercase letters (e.g., $\mathbf{x}$) represent vectors, and boldface uppercase letters (e.g., $\mathbf{X}$) represent matrices. Functions with uppercase letters (e.g., $X(f)$) denote Fourier transforms, and $*$ denotes convolution integral. The trace of a matrix is denoted by $\mathrm{tr}\{\cdot\}$, the transpose

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is denoted by $\{\cdot\}^{\mathcal{T}}$, while the conjugate transpose is denoted by $\{\cdot\}^{\mathcal{H}}$.

II. System Model for SC LMMSE-FDE

In this section, we present the system model for the SC LMMSE-FDE. Information bearing symbols $\{a[n]\}_n$ are modeled as independent and identically distributed (i.i.d.) proper-complex random variables with mean zero and variance σ_s^2. This sequence of data symbols is blocked into vectors $\{\mathbf{a}_l\}$ of length N, where

$$\mathbf{a}_l \triangleq \left[a[lN], a[lN+1], ..., a[lN+N-1]\right]^{\mathcal{T}}. \tag{1}$$

In order to avoid interblock interference (IBI), a CP of length K is appended. The resulting sequence $\{a_c[n]\}_n$ is linearly modulated with transmit pulse $g_T(t)$ and symbol rate $1/T$. The complex envelope of the transmitted signal is given by

$$x(t) = \sum_{n=-\infty}^{\infty} a_c[n] g_T(t - nT). \tag{2}$$

This signal passes through a frequency selective channel and is fed to a receive filter. When the impulse response of the channel and the receive filter are denoted by $h(t)$ and $g_R(t)$, respectively, the output of the receive filter can be written as

$$y(t) = \sum_{n=-\infty}^{\infty} a_c[n] p(t - nT) + v(t). \tag{3}$$

In (3), $p(t)$ represents the overall channel defined as $p(t) \triangleq g_T(t) * h(t) * g_R(t)$, i.e.,

$$P(f) = G_T(f) H(f) G_R(f) \tag{4}$$

where $G_T(f), H(f)$, and $G_R(f)$ are the Fourier transforms of $g_T(t), h(t)$, and $g_R(t)$, respectively. The noise component $v(t)$, which is in general a colored noise, is defined as $v(t) \triangleq n(t) * g_R(t)$, where $n(t)$ is a proper-complex Gaussian random process with two-sided power spectral density $2N_0$.

To produce discrete-time signal, $y(t)$ is sampled at a rate of M/T, where the over-sampling factor M is assumed to be an integer. The receive filter is assumed to be a square-root Nyquist filter that produces white noise when sampled at the rate M/T, i.e.,

$$\sum_{k=-\infty}^{\infty} \left| G_R\left(f + \frac{M}{T}k\right)\right|^2 = 1. \tag{5}$$

After KM samples that correspond to the CP of the first arriving path are removed, the samples are again blocked into vectors $\{\mathbf{b}_l\}_l$ of length MN. We assume that the sampled overall-channel $\{p[n]\}_n \triangleq p(nT/M)$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} \left|p[n]\right|^2 < \infty \tag{6}$$

and has a finite impulse response of order L, i.e., $\{p[n]\}_n$ is zero except on the interval $n \in [c, c+L-1]$ for some integer c. Thus, if the length K of the CP is chosen to satisfy $L \leq KM + 1$, then the IBI is completely eliminated and each block can be handled separately.

We consider only the zeroth received block without loss of generality, and drop the block index for notational simplicity. Then, it can be shown that the $MN \times 1$ vector $\mathbf{b}$ of received symbols is given by

$$\mathbf{b} = \mathbf{P}\mathbf{T}^{(M)}\mathbf{a} + \mathbf{v} \tag{7}$$

where $MN \times MN$ circulant matrix $\mathbf{P}$ represents the overall channel with first column equal to $\left[p[c], p[c+1], ..., p[c+L-1], 0, ..., 0\right]^{\mathcal{T}}$, $MN \times 1$ vector $\mathbf{v}$ has the entries that are i.i.d. proper complex random variables with mean zero and variance $2MN_0/T$, and $MN \times N$ matrix $\mathbf{T}^{(M)}$ inserts $M-1$ zeros after each entry of $\mathbf{a}$, e.g.,

$$\mathbf{T}^{(2)}\mathbf{a} = \left[a[0], 0, a[1], 0, a[2], \ldots, a[N-1], 0\right]^{\mathcal{T}}. \tag{8}$$

For the frequency domain equalization, each block $\mathbf{b}$ is first transformed to the frequency domain, pre-multiplied by an equalization matrix, and then transformed back to the time domain. Thus, the estimate of the data block $\mathbf{a}$ is generated by

$$\hat{\mathbf{a}} = \mathbf{U}_N^{-1}\mathbf{W}\mathbf{U}_{MN}\mathbf{b} \tag{9}$$

where $\mathbf{U}_N$ and $\mathbf{U}_{MN}$ are the $N \times N$ and the $MN \times MN$ discrete Fourier transform (DFT) matrices, and the $N \times MN$ equalization matrix $\mathbf{W}$ is chosen to minimize the MSE defined as

$$\epsilon \triangleq \mathbb{E}\{\|\mathbf{a} - \hat{\mathbf{a}}\|^2\} \tag{10}$$

where $\|\cdot\|$ denotes the vector norm.

Theorem 1: The LMMSE equalization matrix is given by $\mathbf{W}_{\text{LMMSE}} = \left[\mathbf{W}_0 \quad \mathbf{W}_1 \quad \cdots \mathbf{W}_{M-1}\right]$, where $N \times N$ diagonal matrix $\mathbf{W}_i, i = 0, 1, ..., M-1$, is defined as

$$\mathbf{W}_i \triangleq \text{diag}\Bigg\{ \frac{P[iN]^*}{\frac{2M^2N_o}{T\sigma_s^2} + \sum_{k=0}^{M-1}|P[kN]|^2}, \ldots, \frac{P[iN+N-1]^*}{\frac{2M^2N_o}{T\sigma_s^2} + \sum_{k=0}^{M-1}|P[kN+N-1]|^2} \Bigg\} \tag{11}$$

and where $P[n]$ is the MN-point DFT of the first column of $\mathbf{P}$. In addition, the corresponding MSE, denoted by $\epsilon_{\text{FDE}}(M, N)$ is given by

$$\epsilon_{\text{FDE}}(M, N) = \frac{1}{N}\sum_{l=0}^{N-1} \frac{\sigma_s^2 N_0}{\frac{T\sigma_s^2}{2M^2}\sum_{k=0}^{M-1}|P[l+kN]|^2 + N_0}. \tag{12}$$

Proof: See [6].[1] □

For a fair comparison of the SC LMMSE-FDE to the CT LMMSE receiver, the power loss caused by appending the CP must be taken into account. In the SC block transmission with CP, a fraction $\frac{K}{N+K}$ of the average power is consumed in the CP. By replacing the transmit pulse $g_T(t)$ in the above

[1] In [6], the forward filter coefficients that minimize the MSE of SC FDE with decision feedback are derived. From this result, the conclusions can be easily derived.

derivations with $\sqrt{\frac{N}{N+K}}g_T(t)$, we can make the average transmit power equal to that of the conventional SC scheme having the average symbol energy σ_s^2 and the transmit pulse $g_T(t)$. The CP also consumes $\frac{K}{N+K}$ of the total transmission time. However, as far as the length $K(N)$ of the CP as a function of the block length N satisfies the condition

$$\lim_{N\to\infty}\frac{K(N)}{N}=0 \tag{13}$$

the loss in the transmission rate becomes zero as N tends to infinity, which is the case with our asymptotic analysis presented in the next section.

III. Asymptotic Equivalence

In this section, the asymptotic equivalence of the SC LMMSE-FDE to the CT LMMSE equalizer is established for band-limited systems. Since a strictly band-limited $p(t)$ has an infinite duration in the time domain, the IBI always exists though its effect vanishes, unless we allow the length K of CP to increase unboundedly as N tends to infinity. In what follows, we allow

$$\lim_{N\to\infty}K(N)=\infty \tag{14}$$

but still make $K(N)$ satisfy the condition (13). Thus, there is no loss in the transmission rate but the vanishing effect of the IBI is systematically handled in the asymptotic analysis for band-limited systems. A simple example of $K(N)$ that satisfies the conditions (13) and (14) is $K(N)=\lceil\sqrt{N}\rceil$.

Corollary 1: If the overall channel response longer than the length of CP is ignored, the MSE $\hat{\epsilon}_{\text{FDE}}(M,N)$ of the band-limited SC LMMSE-FDE is given by

$$\hat{\epsilon}_{\text{FDE}}(M,N)=\frac{1}{N}\sum_{l=0}^{N-1}\frac{\sigma_s^2N_0}{\frac{T\sigma_s^2}{2M^2}\sum_{k=0}^{M-1}|\hat{P}[l+kN]|^2+N_0} \tag{15}$$

where $\{\hat{P}[l]\}_{l=0}^{MN}$ is the MN-point DFT of

$$\hat{p}[n]\triangleq p[n]\big(u[n+M\lfloor K/2\rfloor]-u[n-M\lceil K/2\rceil]\big). \tag{16}$$

Proof: Since the tails of $\{p[n]\}_n$ are ignored, the resulting overall channel $\hat{p}[n]$ is given by (16), where $u[n+M\lfloor K/2\rfloor]-u[n-M\lceil K/2\rceil]$ is the windowing function multiplied to $\{p[n]\}_n$. So, the first column of $\mathbf{P}$ in (7) now becomes $\{\hat{p}[n]\}_n$ viewed through the window. Then, by replacing the original sampled overall-channel $\{p[n]\}_n$ with the windowed sampled overall-channel $\{\hat{p}[n]\}_n$, we can immediately obtain the MSE (15) of the SC LMMSE-FDE from *Theorem 1*. □

Now, we derive the asymptotic MSE of the band-limited SC LMMSE-FDE.

Theorem 2: The MSE $\hat{\epsilon}_{\text{FDE}}(M,N)$ of the band-limited SC LMMSE-FDE converges to $\hat{\epsilon}_{\text{FDE}}(M)$ defined as

$$\hat{\epsilon}_{\text{FDE}}(M)\triangleq \int_{-\frac{1}{2T}}^{\frac{1}{2T}}\frac{T\sigma_s^2N_0}{\frac{\sigma_s^2}{2T}\sum_{k=0}^{M-1}\left|\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m+\frac{k}{T})\right|^2+N_o}df \tag{17}$$

as $N\to\infty$.

Proof: The first step is to represent $\hat{P}[l]$ in (15) in terms of $P(f)$ and redefine it as $\frac{M}{T}Q_{M,N}(\frac{l}{NT})$, i.e.,

$$\begin{aligned}\hat{P}[l]&=\frac{M}{T}\int_0^1\sum_{m=-\infty}^{\infty}P\left(\frac{l}{NT}-\frac{M}{T}f'+\frac{M}{T}m\right)\\&\quad\times e^{j\pi f'}\frac{e^{2\pi M\lfloor K(N)\rfloor f'/2}-e^{-j2\pi M\lceil K(N)\rceil f'/2}}{\sin\pi f'}df'\\&\triangleq\frac{M}{T}Q_{M,N}\left(\frac{l}{NT}\right).\end{aligned} \tag{18}$$

Substituting (18) into (12) leads to

$$\begin{aligned}&\hat{\epsilon}_{\text{FDE}}(M,N)\\&=\frac{1}{N}\sum_{l=0}^{N-1}\frac{\sigma_s^2N_0}{\frac{\sigma_s^2}{2T}\sum_{k=0}^{M-1}|Q_{M,N}\left(\frac{l}{NT}+\frac{k}{T}\right)|^2+N_0}\end{aligned} \tag{19}$$

where $Q_{M,N}(f)$ can be rewritten as

$$Q_{M,N}(f)=\frac{T}{M}\sum_{n=-M\lfloor K(N)\rfloor/2}^{M\lceil K(N)\rceil/2-1}p[n]e^{-j2\pi\frac{T}{M}fn}. \tag{20}$$

Note that (20) takes the form of discrete-time Fourier transform (DTFT). When $p[n]$ is absolutely summable as (6), it is well-known that the right side of (20) converges uniformly to its DTFT output $\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m)$ as $N\to\infty$ [8]. Note that $\sup_f(|Q_{M,N}(f)|+|\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m)|)$ is bounded because it is the supremum of the sum of two continuous functions. Then, by *Lemma 1* in the Appendix, we can show that $|Q_{M,N}(f)|^2$ converges uniformly to $|\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m)|^2$ as $N\to\infty$. Thus, by *Lemma 2* in the Appendix, the summand in (19) also converges uniformly to

$$\frac{\sigma_s^2N_0}{\frac{\sigma_s^2}{2T}\sum_{k=0}^{M-1}|\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m+\frac{k}{T})|^2+N_0} \tag{21}$$

as $N\to\infty$, because the denominators in (19) and (21) are both bounded below by N_0. Finally, by *Lemma 3* in the Appendix, this uniform convergence to (21) guarantees that $\hat{\epsilon}_{\text{FDE}}(M,N)$ in (19) converges to

$$\int_0^{\frac{1}{T}}\frac{T\sigma_s^2N_o}{\frac{\sigma_s^2}{2T}\sum_{k=0}^{M-1}\left|\sum_{m=-\infty}^{\infty}P(f+\frac{M}{T}m+\frac{k}{T})\right|^2+N_o}df. \tag{22}$$

as $N\to\infty$. Note that the folded spectrum in the denominator is periodic with period $1/T$. Therefore, by the change of variable, we obtain (17). □

As easily seen in (17), the asymptotic MSE depends on the receive filter through $P(f)$. In the following theorem, we derive the optimum receive filter that minimizes $\hat{\epsilon}_{\text{FDE}}(M)$.

Theorem 3: If $P(f)$ is band-limited to $|f|<D/T$, then the impulse response of the optimum receive filter that is the solution to

$$\min_{G_R(f)}\ \hat{\epsilon}_{\text{FDE}}(M)$$
$$\text{subject to}\ \sum_{k=-\infty}^{\infty}\left|G_R\left(f+\frac{M}{T}k\right)\right|^2=1 \tag{23}$$

is given by

$$\hat{G}_R(f) = \frac{G_T(f)^* H(f)^*}{\sqrt{\sum_{m=-\infty}^{\infty} \left| G_T\left(f + \frac{M}{T}m\right) H\left(f + \frac{M}{T}m\right)\right|^2}} \tag{24}$$

for $|f| < D/T$, which is the whitened matched filter [11] for the sampling rate M/T. Moreover, the minimized MSE is given by

$$\int_0^{\frac{1}{T}} \frac{T\sigma_s^2 N_0}{\frac{\sigma_s^2}{2T}\sum_{k=-\infty}^{\infty}\left|G_T\left(f+\frac{k}{T}\right)H\left(f+\frac{k}{T}\right)\right|^2 + N_0} df. \tag{25}$$

Proof: As mentioned in the previous section, $g_R(t)$ is restricted to a square-root Nyquist pulse to obtain white sampled-noise. The optimization problem (23) is tackled by using the vectorized Fourier transform (VFT) [7]. First, we define the VFTs $(\mathbf{g}_T * \mathbf{h})(f)$ and $\mathbf{g}_R(f)$ of $g_T(t) * h(t)$ and $g_R(t)$, respectively, as

$$(\mathbf{g}_T * \mathbf{h})(f) \triangleq \begin{bmatrix} G_T\left(-\frac{D}{T}+f\right)H\left(-\frac{D}{T}+f\right) \\ G_T\left(-\frac{D-1}{T}f\right)H\left(-\frac{D-1}{T}+f\right) \\ \vdots \\ G_T\left(\frac{D}{T}+f\right)H\left(\frac{D}{T}+f\right) \end{bmatrix} \tag{26}$$

and

$$\mathbf{g}_R(f) \triangleq \begin{bmatrix} G_R(-\frac{D}{T}+f) \\ G_R(-\frac{D-1}{T}+f) \\ \vdots \\ G_R(\frac{D}{T}+f) \end{bmatrix} \tag{27}$$

for $f \in [-1/(2T), 1/(2T))$. Next, we define $\mathbf{g}_R^{(k)}(f)$, for $k = 0, 1, ..., M-1$, as a length-$(2D+1)$ vector that is the collection of all jth entries of $\mathbf{g}_R(f)$ such that $\mathrm{mod}(j-1, M) = k$, e.g., if $M = 2$ and $D = 1$, then

$$\mathbf{g}_R^{(0)}(f) = \begin{bmatrix} G_R(-\frac{1}{T}+f) \\ 0 \\ G_R(\frac{1}{T}+f) \end{bmatrix} \quad \text{and} \quad \mathbf{g}_R^{(1)}(f) = \begin{bmatrix} 0 \\ G_R(\frac{1}{T}+f) \\ 0 \end{bmatrix}. \tag{28}$$

Finally, we define $(\mathbf{g}_T * \mathbf{h})^{(k)}(f)$ in the same way. Now, the problem (23) can be rewritten as

$$\min_{\mathbf{g}_R(f)} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{T\sigma_s^2 N_0}{\frac{\sigma_s^2}{2T}\sum_{k=0}^{M-1}|(\mathbf{g}_T * \mathbf{h})^{(k)}(f)^T \mathbf{g}_R^{(k)}(f)|^2 + N_0} df \tag{29}$$

$$\text{subject to } \|\mathbf{g}_R^{(k)}(f)\|^2 = 1, \text{for } k = 0, 1, ..., M-1. \tag{30}$$

Since the integrand of the objective function in (29) is positive, we can minimize (29) by minimizing the integrand for each $f \in [-1/(2T), 1/(2T))$. Moreover, the integrand is minimized if we maximize $|(\mathbf{g}_T * \mathbf{h})^{(k)}(f)^T \mathbf{g}_R^{(k)}(f)|^2$ for each k. This is a matched filtering problem, and $\hat{\mathbf{g}}_R^{(k)}(f)$ that maximizes the summand in the denominator of (29) subject to the constraint (30) is given by

$$\hat{\mathbf{g}}_R^{(k)}(f) = \frac{(\mathbf{g}_T * \mathbf{h})^{(k)}(f)^*}{\|(\mathbf{g}_T * \mathbf{h})^{(k)}(f)\|}. \tag{31}$$

Now we can construct the optimum $\hat{\mathbf{g}}_R(f)$ by simply combining $\hat{\mathbf{g}}_R^{(k)}(f)$ for all k. Then, by transforming $\hat{\mathbf{g}}_R(f)$ to $\hat{G}_R(f)$, we obtain (24) and, consequently, (25). □

This theorem specializes to the following corollary.

Corollary 2: For $M \geq 2D$, i.e., the received signal is sampled faster than the Nyquist rate, any square-root Nyquist pulse for the sampling rate M/T, which has flat energy spectral density on the support of $G_T(f)H(f)$, is the optimum response $\hat{G}_R(f)$. For $M < 2D$, $\hat{G}_R(f)$ is determined by $G_T(f)H(f)$ at every f on the support of $G_T(f)H(f)$.

Proof: When $M \geq 2D$, $\hat{G}_R(f)$ reduces to

$$\hat{G}_R(f) = e^{\mathrm{j}\angle G_T(f)^* H(f)^*}. \tag{32}$$

for the support of $G_T(f)H(f)$, and to an arbitrary waveform elsewhere as far as the constraint in (23) is satisfied. This is because on the support no folding actually occurs in the folded spectrum, so that the denominator is cancelled by the square-root of the numerator with an arbitrary phase response. When $M < 2D$, no cancellation occurs in any frequency, so that $\hat{G}_R(f)$ depends on $G_T(f)H(f)$ for every f. Therefore, the conclusion follows. □

The optimum asymptotic MSE given by (25) turns out to be the MSE of the CT LMMSE receiver [12]. So, *Theorem 3* and *Corollary 2* provides the necessary and sufficient condition for the asymptotic equivalence of the SC LMMSE-FDE to the CT LMMSE equalizer in the MSE sense: For the asymptotic equivalence, it is necessary and sufficient to use a whitened matched filter. Note that this condition cannot be satisfied by a fixed receive filter unless $M \geq 2D$, i.e., the received signal is sampled faster than the Nyquist rate. For example, the SC modulation with excess bandwidth less than or equal to one, i.e., $D = 1$, has no fixed receive filter followed by a symbol-rate sampler as the optimal receive filter for all channel response. If $M \geq 2D$, then any square-root Nyquist pulse having flat spectrum on the support of $G_T(f)H(f)$ is optimal because it can be viewed as a whitened matched filter.

IV. Numerical Results

We conduct computer simulations to support the theoretical results. The signal-to-interference plus noise ratio (SINR) performance of the SC LMMSE-FDE is compared with that of the CT LMMSE equalizer for various block lengths. The length of CP is $K(N) = \lceil\sqrt{N}\rceil$, and the transmit pulse $g_T(t)$ is the square-root raised cosine pulse with roll-off factor 0.2. The impulse response of the channel is $h(t) = \sum_{i=0}^{2} c_i\delta(t - d_iT)$, where $[c_0, c_1, c_2] = [-2.9487 + \mathrm{j}2.8957,\ 1.2827 + \mathrm{j}1.9531,\ 0.2590 + \mathrm{j}2.4098]$ and $[d_0, d_1, d_2] = [0.0,\ 5.4,\ 9.3]$. The receive pulse $g_R(t)$ is the square-root raised cosine pulse when $M = 1$, and the ideal low pass filter (LPF) when $M = 2$. The results for $M > 2$ are omitted because the difference in SINR from the case with $M = 2$ is negligible.

In Figs. 1 and 2, the matched filter bound (MFB) [10], the SINR of the CT LMMSE equalizer, and the SINRs of the SC LMMSE-FDE with $M = 1$ and $M = 2$ are illustrated for small to medium and large block lengths, respectively. The SINR of the SC LMMSE-FDE is obtained from the MSE that

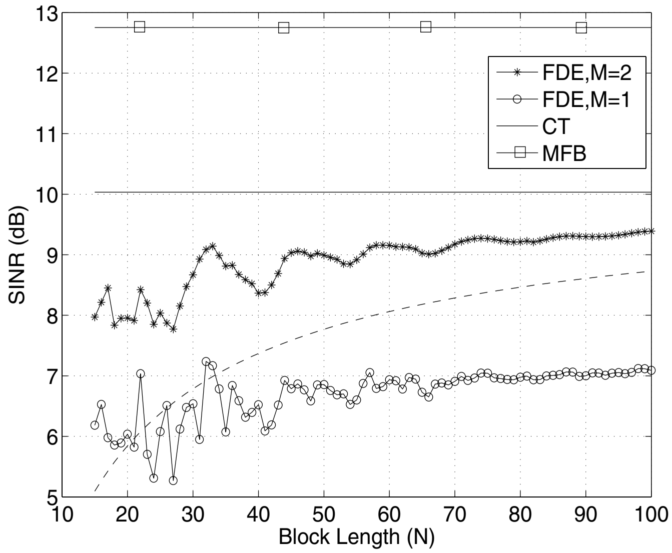

Fig. 1. SINR performance of the SC LMMSE-FDE and CT LMMSE equalizer for small to medium block lengths.

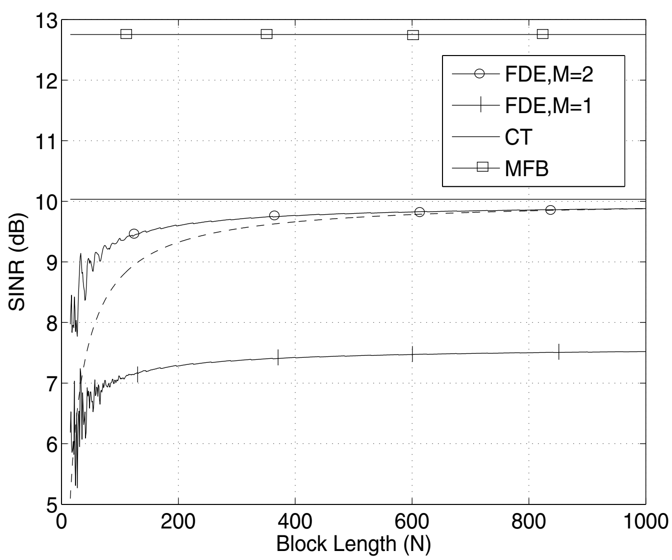

Fig. 2. SINR performance of the SC LMMSE-FDE and CT LMMSE equalizer for large block lengths.

is evaluated from (15). The dashed lines represent $\frac{N}{N+L}$ times the SINR of the CT LMMSE equalizer. Notice that the SINRs of the SC LMMSE-FDE are not monotonically increasing in N but fluctuate for small to medium block lengths. On the contrary, the SINRs are almost monotonically increasing in N for large block lengths. Especially for $M = 2$ and large N, it converges to that of the CT LMMSE equalizer by following the dashed line, which means the convergence rate is approximately $\frac{N}{N+L}$. However, for $M = 1$, the fixed receive filter does not asymptotically achieve the MSE of the CT LMMSE equalizer as predicted in *Corollary 2*.

Fig. 3 that depicts the magnitude response of the overall channel $p[n]$ is shown to explain the fluctuation of the SINR.

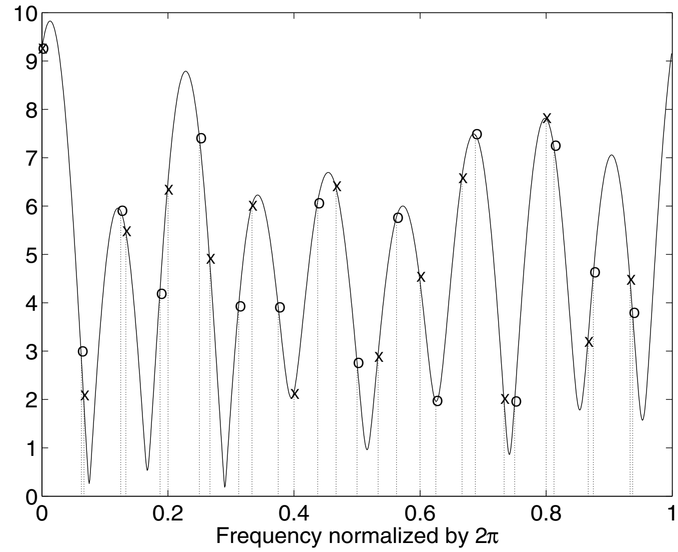

Fig. 3. Frequency response of overall channel $p[n]$ and the samples for $N = 15$ and $N = 16$

Since the SINR and the MSE are related as $\text{SINR} = \frac{1}{\text{MSE}} - 1$, this fluctuation can be explained by the fluctuation in the MSE. The sampling points marked by x represent the points used to evaluate (12) for $N = 15$, which are equal to the 15-point DFT of $p[n]$, while the sampling points marked by o represent the points used to evaluate (12) for $N = 16$, which are equal to the 16-point DFT of $p[n]$. Note that, roughly speaking, the MSE becomes smaller when these samples have larger values. However, a larger N may result in the sampling locations that have smaller values. This effect, of course, will vanish and the MSE will converge to that of the CT LMMSE-FDE as N tends to infinity because more and more samples are densely taken. This is why the SINR convergence rate becomes $\frac{N}{N+L}$, which only reflects the effect of bit energy normalization.

V. Conclusions

In this paper, we have formally proved the asymptotic equivalence of the SC LMMSE-FDE to the CT LMMSE equalizer in the limit of a very large transmission block length. The MSE of the SC LMMSE-FDE is derived and it is shown to converge to that of the CT LMMSE equalizer under certain conditions. The optimum receive filter of the SC LMMSE-FDE is also derived to identify the necessary and sufficient condition for the equivalence. It is shown that a receive filter fixed to a function that has a flat energy spectral density over the support of frequency response of transmit pulse can achieves the asymptotic MSE when the received signal is sampled faster than the Nyquist rate.

Appendix

In this Appendix, we provide three lemmas used to prove *Theorem 2* in Section III. The common assumption is that the sequence of functions $f_N(x),\ N = 1, 2, 3, ...$ converges uniformly [9] on $x \in [0, 1]$ to a function $f(x)$, i.e., $\varepsilon_N \to 0$

as $N \to \infty$, where

$$\varepsilon_N = \sup_x |f_N(x) - f(x)|. \tag{33}$$

Lemma 1: If $\sup_x (|f_N(x)| + |f(x)|)$ is bounded, then $|f_N(x)|^2$ converges to $|f(x)|^2$ uniformly.

Proof: Define $\hat{\varepsilon}_N$ as $\hat{\varepsilon}_N \triangleq \sup_x \left| |f_N(x)|^2 - |f(x)|^2 \right|$. Then,

$$\begin{aligned} \hat{\varepsilon}_N &= \sup_x (|f_N(x)| + |f(x)|)(|f_N(x)| - |f(x)|) &(34)\\ &\le \sup_x (|f_N(x)| + |f(x)|) \sup_x (|f_N(x)| - |f(x)|) &(35)\\ &\le \sup_x (|f_N(x)| + |f(x)|) \sup_x (f_N(x) - f(x)) &(36)\\ &\le \sup_x (|f_N(x)| + |f(x)|)\, \varepsilon_N. &(37) \end{aligned}$$

Thus, by the assumption, $\hat{\varepsilon}_N \to 0$ as $N \to 0$. Therefore, the conclusion follows. □

Lemma 2: If $|f(x)|$ and $|f_N(x)|$ are bounded below by positive numbers, then $\frac{1}{f_N(x)}$ converges to $\frac{1}{f(x)}$ uniformly.

Proof: Define $\hat{\varepsilon}_N$ as $\hat{\varepsilon}_N \triangleq \sup_x \left| \frac{1}{f_N(x)} - \frac{1}{f(x)} \right|$. Then,

$$\hat{\varepsilon}_N = \sup_x \left| \frac{f_N(x) - f(x)}{f(x) f_N(x)} \right| \le \frac{\sup_x |f_N(x) - f(x)|}{\inf_x |f(x) f_N(x)|} \le \frac{\varepsilon_N}{ab} \tag{38}$$

where $a = \inf_x |f(x)| > 0$ and $b = \inf_x |f_N(x)| > 0$ by the assumption. Thus, ab is finite. Therefore, the conclusion follows, because $\hat{\varepsilon}_N \to 0$ as $N \to \infty$. □

Lemma 3: $\frac{1}{N} \sum_{n=1}^{N} f_N\left(\frac{n}{N}\right)$ converges to $\int_0^1 f(x) dx$.

Proof: Define $\hat{\varepsilon}_N$ as $\varepsilon_N \triangleq \sup_{x \in [0,1]} |f(x) - f_N(x)|$. Then, $f(x) - \varepsilon_N \le f_N(x) \le f(x) + \varepsilon_N, \forall x \in [0, 1]$. By evaluating at $x = 1/N,\ 2/N,\ \ldots\ N/N$ and taking the average, we obtain

$$\frac{1}{N} \sum_{n=1}^{N} f\left(\frac{n}{N}\right) - \varepsilon_N \le \frac{1}{N} \sum_{n=1}^{N} f_N\left(\frac{n}{N}\right) \le \frac{1}{N} \sum_{n=1}^{N} f\left(\frac{n}{N}\right) + \varepsilon_N. \tag{39}$$

If we let $N \to \infty$, then $\varepsilon_N \to 0$ leads to

$$\int_0^1 f(x) dx \le \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_N\left(\frac{n}{N}\right) \le \int_0^1 f(x) dx \tag{40}$$

Therefore, the conclusion follows. □

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