

Name:

Hemos ID:

CSE-433 Logic in Computer Science 2010 Final exam — Sample Solution

- This is a closed-book exam. No other material is permitted.
- It consists of 5 problems worth a total of 210 points.
- There are 7 pages in this exam, including 3 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Total
Score						
Max	30	30	50	50	50	210

1 Definitional equality and terms [30pts]

We write $t = s$ to mean that terms t and s are definitionally equal.

$$\frac{t \Longrightarrow_{\beta}^* r \quad s \Longrightarrow_{\beta}^* r}{t = s} \text{DefEq1}$$

In the introduction rule *DefEq1*, the judgment $t \Longrightarrow_{\beta}^* r$ means that t reduces to r by zero or more β -reductions. Here we assume that β -reductions may be applied to subterms of the term being reduced.

Question 1. [5 pts] Definitional equality is transitive. Yes or no? Explain why. (-5 pts for a wrong answer)
No

Question 2. [5 pts] Consider terms for datatype nat:

$$\text{term } t ::= \dots \mid \mathbf{0} \mid \mathbf{s}(t) \mid \text{rec } f(t) \text{ of } f(\mathbf{0}) \Rightarrow t \mid f(\mathbf{s}(x)) \Rightarrow t$$

Write two β -reductions for datatype nat:

$$\begin{array}{l} \text{rec } f(\mathbf{0}) \text{ of } f(\mathbf{0}) \Rightarrow t_0 \mid f(\mathbf{s}(x)) \Rightarrow t_s \quad \Longrightarrow_{\beta} \quad t_0 \\ \text{rec } f(\mathbf{s}(t)) \text{ of } f(\mathbf{0}) \Rightarrow t_0 \mid f(\mathbf{s}(x)) \Rightarrow t_s \quad \Longrightarrow_{\beta} \quad [\text{rec } f(t) \text{ of } f(\mathbf{0}) \Rightarrow t_0 \mid f(\mathbf{s}(x)) \Rightarrow t_s / f(x)][t/x]t_s \end{array}$$

Question 3. [5 pts] Define a function *plus* of datatype $\text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ adding two natural numbers.

$$\text{plus} = \lambda x \in \text{nat}. \lambda y \in \text{nat}. \text{rec } p(x) \text{ of } p(\mathbf{0}) \Rightarrow y \mid p(\mathbf{s}(z)) \Rightarrow \mathbf{s}(p(z))$$

Question 4. [5 pts] *plus 0 x = x* holds. True or false? (-5 pts for a wrong answer)
True.

Question 5. [5 pts] *plus x 0 = x* holds. True or false? (-5 pts for a wrong answer)
False.

Question 6. [5 pts] *plus s(x) y = s(plus x y)* holds. True or false? (-5 pts for a wrong answer)
True.

2 Local and global soundness and completeness [30pts]

Question 1. [5 pts] Explain local soundness in no more than 100 words.

If a detour involving a specific connective can be transformed to a proof of the same judgment with no detour, the inference rules for the connective are locally sound. It is a property of a specific connective (or its inference rules).

Question 2. [10 pts] Explain global soundness in no more than 100 words.

If any proof can be transformed to another proof of the same judgment with no detour, the system of interference rules is globally sound. It is not a property of a specific connective (or its inference rules).

Question 3. [5 pts] Explain local completeness in no more than 100 words.

If a proof involving a specific connective can be expanded to a proof of the same judgment in which the connective is eliminated, the inference rules for the connective are locally complete. It is a property of a specific connective (or its inference rules).

Question 4. [10 pts] Explain global completeness in no more than 100 words.

If a proof can be expanded to a proof of the same judgment in which all the connective in it are eventually eliminated, the system of inference rules is globally complete. It is not a property of a specific connective (or its inference rules).

3 First-order logic [50 pts]

Consider the natural deduction system for first-order logic (without datatypes). We use t for terms, x for term variables, and a for parameters.

$$\frac{[a/x]A \text{ true}}{\forall x.A \text{ true}} \forall I^a \quad \frac{\forall x.A \text{ true}}{[t/x]A \text{ true}} \forall E \quad \frac{[t/x]A \text{ true}}{\exists x.A \text{ true}} \exists I \quad \frac{\exists x.A \text{ true} \quad \overline{[a/x]A \text{ true}}^w \quad \vdots \quad C \text{ true}}{C \text{ true}} \exists E^{a,w}$$

Question 1. [10 pts]

Propose proof terms and their typing rules for first-order logic. Do not use hypothetical judgments with contexts.

$$\frac{[a/x]M : [a/x]A}{\lambda x.M : \forall x.A} \forall I^a \quad \frac{M : \forall x.A}{M t : [t/x]A} \forall E \quad \frac{M : [t/x]A}{\langle t, M \rangle : \exists x.A} \exists I \quad \frac{M : \exists x.A \quad \overline{[a/x]N : C} \quad \vdots \quad \text{let } \langle x, w \rangle = M \text{ in } N : C}{\text{let } \langle x, w \rangle = M \text{ in } N : C} \exists E^a$$

Question 2. [10 pts]

Propose four inference rules for normal and neutral judgments ($A \uparrow$ and $A \downarrow$) in the natural deduction style. Do not use hypothetical judgments with contexts.

$$\frac{[a/x]A \uparrow}{\forall x.A \uparrow} \forall I_\uparrow^a \quad \frac{\forall x.A \downarrow}{[t/x]A \downarrow} \forall E_\downarrow \quad \frac{[t/x]A \uparrow}{\exists x.A \uparrow} \exists I_\uparrow \quad \frac{\exists x.A \downarrow \quad \overline{[a/x]A \downarrow}^w \quad \vdots \quad C \uparrow}{C \uparrow} \exists E_\uparrow^{a,w}$$

For each case below, complete the CPS translation M° of a given proof term M . Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\mathcal{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_1 M^\circ : \neg\neg C^\circ$ where*

$$\begin{aligned}\Gamma^\circ &= \{x : A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ &= \{x : \neg A^\circ \mid x : A \text{ false} \in \Delta\}.\end{aligned}$$

Case $\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathcal{K}} x : A}$ Hyp

$$x^\circ = \lambda k : A^\circ \supset \perp. k \ x$$

Case $\frac{\Gamma, x : A; \Delta \vdash_{\mathcal{K}} M : B}{\Gamma; \Delta \vdash_{\mathcal{K}} \lambda x : A. M : A \supset B}$ $\supset I$

$$(\lambda x : A. M)^\circ = \lambda k : (A^\circ \supset \neg\neg B^\circ) \supset \perp. k \ (\lambda x : A^\circ. M^\circ)$$

Case $\frac{\Gamma; \Delta \vdash_{\mathcal{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathcal{K}} N : A}{\Gamma; \Delta \vdash_{\mathcal{K}} M \ N : B}$ $\supset E$

$$(M \ N)^\circ = \lambda k : B^\circ \supset \perp. M^\circ \ (\lambda x : A^\circ \supset \neg\neg B^\circ. N^\circ \ (\lambda y : A^\circ. x \ y \ k))$$

Case $\frac{\Gamma; \Delta, x : A \text{ false} \vdash_{\mathcal{K}} M : A}{\Gamma; \Delta \vdash_{\mathcal{K}} \text{callcc } x : A \text{ false}. M : A}$ Callcc

$$(\text{callcc } x : A \text{ false}. M)^\circ = \lambda k : A^\circ \supset \perp. [k/x]M^\circ \ k$$

Case $\frac{\Gamma; \Delta \vdash_{\mathcal{K}} M : A \quad x : A \text{ false} \in \Delta}{\Gamma; \Delta \vdash_{\mathcal{K}} \text{throw } M \ \text{to } x : C}$ Throw

$$(\text{throw } M \ \text{to } x)^\circ = \lambda k : C^\circ \supset \perp. M^\circ \ x$$

Question 3. [20 pts] The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M \ N$ specifies that we finish evaluating N before we apply the function from M to the result of evaluating N .

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both A° and B° :

$$\begin{aligned} P^\circ &= P \\ (A \supset B)^\circ &= \neg\neg A^\circ \supset \neg\neg B^\circ \end{aligned}$$

For each case below, complete the CPS translation M° of a given proof term M . Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\text{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\text{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{aligned} \Gamma^\circ &= \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ &= \{x : \neg A^\circ \mid x : A \text{ false} \in \Delta\}. \end{aligned}$$

Note the change in the definition of Γ° which now assigns to x type $\neg\neg A^\circ$. The translation of $\text{callcc } x : A \text{ false}. M$ and $\text{throw } M \text{ to } x$ is the same as in the previous CPS translation and is omitted.

Case $\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\text{K}} x : A}$ Hyp

$$x^\circ = \lambda k : A^\circ \supset \perp. x k$$

Case $\frac{\Gamma, x : A; \Delta \vdash_{\text{K}} M : B}{\Gamma; \Delta \vdash_{\text{K}} \lambda x : A. M : A \supset B}$ $\supset\text{I}$

$$(\lambda x : A. M)^\circ = \lambda k : (\neg\neg A^\circ \supset \neg\neg B^\circ) \supset \perp. k (\lambda x : \neg\neg A^\circ. M^\circ)$$

Case $\frac{\Gamma; \Delta \vdash_{\text{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\text{K}} N : A}{\Gamma; \Delta \vdash_{\text{K}} M N : B}$ $\supset\text{E}$

$$(M N)^\circ = \lambda k : B^\circ \supset \perp. M^\circ (\lambda x : \neg\neg A^\circ \supset \neg\neg B^\circ. x N^\circ k)$$

5 Cut elimination [50 pts]

Consider the following fragment of the sequent calculus for propositional logic (with implication only):

$$\frac{}{\Gamma, A \longrightarrow A} \textit{Init} \quad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R$$

Give a proof of the admissibility of the cut rule:

Theorem (Admissibility of the cut rule). *If $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$, then $\Gamma \longrightarrow C$.*

Begin your proof by stating how your nested induction proceeds, *e.g.*, “By nested induction on the structure of 1) \dots ; 2) \dots ; 3) \dots .” Your proof may use the weakening and contraction properties. In individual steps in your proof, please write the conclusion in the left side and the justification in the right side:

conclusion

justification

Your proof should cover all possible cases.