

Name:

Hemos ID:

CSE-433 Logic in Computer Science 2011 Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 200 points.
- There are 10 pages in this exam, including 1 work sheet.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

	Problem 1	Problem 2	Problem 3	Problem 4	Total
Score					
Max	50	50	50	50	200

1 Classical logic [50 pts]

In this problem, we use contexts Γ and Δ defined as follows:

$$\begin{aligned}\Gamma & ::= \cdot \mid \Gamma, x : A \\ \Delta & ::= \cdot \mid \Delta, x : A \text{ false}\end{aligned}$$

We use $\Gamma; \Delta \vdash_{\kappa} C \text{ true}$ for judgments in classical logic, $\Gamma; \Delta \vdash_{\kappa} M : C$ for typechecking proof terms in classical logic, and $\Gamma \vdash_1 M : C$ for typechecking proof terms in constructive logic.

Here are the rules $\text{Contra}\uparrow$ and $\text{Contra}\downarrow$ given in the course notes:

$$\frac{\Gamma; \Delta, A \text{ false} \vdash_{\kappa} A \text{ true}}{\Gamma; \Delta \vdash_{\kappa} A \text{ true}} \text{Contra}\uparrow \qquad \frac{\Gamma; \Delta, A \text{ false} \vdash_{\kappa} A \text{ true}}{\Gamma; \Delta, A \text{ false} \vdash_{\kappa} C \text{ true}} \text{Contra}\downarrow$$

Question 1. [10 pts] Show that the rule Peirce is derivable in classical logic.

$$\frac{}{\cdot; \vdash_{\kappa} ((A \supset B) \supset A) \supset A \text{ true}}$$

Question 2. [20 pts] We use the following double-negation translation for the fragment of propositional logic with implication:

$$\begin{aligned} P^\circ &= P \\ (A \supset B)^\circ &= A^\circ \supset \neg\neg B^\circ \end{aligned}$$

For each case below, complete the CPS translation M° of a given proof term M . Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\mathcal{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_1 M^\circ : \neg\neg C^\circ$ where*

$$\begin{aligned} \Gamma^\circ &= \{x : A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ &= \{x : \neg A^\circ \mid x : A \text{ false} \in \Delta\}. \end{aligned}$$

Case $\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathcal{K}} x : A}$ Hyp

$x^\circ =$

Case $\frac{\Gamma, x : A; \Delta \vdash_{\mathcal{K}} M : B}{\Gamma; \Delta \vdash_{\mathcal{K}} \lambda x : A. M : A \supset B}$ $\supset I$

$(\lambda x : A. M)^\circ =$

Case $\frac{\Gamma; \Delta \vdash_{\mathcal{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathcal{K}} N : A}{\Gamma; \Delta \vdash_{\mathcal{K}} M N : B}$ $\supset E$

$(M N)^\circ =$

Case $\frac{\Gamma; \Delta, x : A \text{ false} \vdash_{\mathcal{K}} M : A}{\Gamma; \Delta \vdash_{\mathcal{K}} \text{callcc } x : A \text{ false}. M : A}$ Callcc

$(\text{callcc } x : A \text{ false}. M)^\circ =$

Case $\frac{\Gamma; \Delta \vdash_{\mathcal{K}} M : A \quad x : A \text{ false} \in \Delta}{\Gamma; \Delta \vdash_{\mathcal{K}} \text{throw } M \text{ to } x : C}$ Throw

$(\text{throw } M \text{ to } x)^\circ =$

Question 3. [20 pts] The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M N$ specifies that we finish evaluating N before we apply the function from M to the result of evaluating N .

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both A° and B° :

$$\begin{aligned} P^\circ &= P \\ (A \supset B)^\circ &= \neg\neg A^\circ \supset \neg\neg B^\circ \end{aligned}$$

For each case below, complete the CPS translation M° of a given proof term M . Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\mathcal{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathcal{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{aligned} \Gamma^\circ &= \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ &= \{x : \neg A^\circ \mid x : A \text{ false} \in \Delta\}. \end{aligned}$$

Note the change in the definition of Γ° which now assigns to x type $\neg\neg A^\circ$. The translation of $\text{callcc } x : A \text{ false. } M$ and $\text{throw } M \text{ to } x$ is the same as in the previous CPS translation and is omitted.

Case $\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathcal{K}} x : A}$ Hyp

$x^\circ =$

Case $\frac{\Gamma, x : A; \Delta \vdash_{\mathcal{K}} M : B}{\Gamma; \Delta \vdash_{\mathcal{K}} \lambda x : A. M : A \supset B}$ $\supset\text{I}$

$(\lambda x : A. M)^\circ =$

Case $\frac{\Gamma; \Delta \vdash_{\mathcal{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathcal{K}} N : A}{\Gamma; \Delta \vdash_{\mathcal{K}} M N : B}$ $\supset\text{E}$

$(M N)^\circ =$

2 Completeness of the sequent calculus [50 pts]

The sequent calculus for propositional logic is given as follows:

$$\begin{array}{c}
 \frac{}{\Gamma, A \rightarrow A} \textit{Init} \quad \frac{\Gamma, A \wedge B, A \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L_L \quad \frac{\Gamma, A \wedge B, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C} \wedge L_R \quad \frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B} \wedge R \\
 \\
 \frac{\Gamma, A \supset B \rightarrow A \quad \Gamma, A \supset B, B \rightarrow C}{\Gamma, A \supset B \rightarrow C} \supset L \quad \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B} \supset R \\
 \\
 \frac{\Gamma, A \vee B, A \rightarrow C \quad \Gamma, A \vee B, B \rightarrow C}{\Gamma, A \vee B \rightarrow C} \vee L \quad \frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B} \vee R_L \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B} \vee R_R \\
 \\
 \frac{}{\Gamma \rightarrow \top} \top R \quad \frac{}{\Gamma, \perp \rightarrow C} \perp L
 \end{array}$$

Show that the following rule $\supset L'$ is incomplete:

$$\frac{\Gamma \rightarrow A \quad \Gamma, A \supset B, B \rightarrow C}{\Gamma, A \supset B \rightarrow C} \supset L'$$

You should present a formula C such $\cdot \rightarrow C$ is provable with the rule $\supset L$, but not with the rule $\supset L'$.

Question 1. [10 pts] What is such a formula C ?

Question 2. [15 pts] Prove $\cdot \rightarrow C$ using the rule $\supset L$.

Question 3. [15 pts] Show that $\cdot \rightarrow C$ is not provable with the rule $\supset L'$.

Question 4. [10 pts] Explain what makes C a counter-example of the completeness of the rule $\supset L'$.

3 Backward proof search [50 pts]

Consider the sequent calculus for backward proof search:

$$\begin{array}{c}
 \frac{\text{atomic } P}{\Gamma, P \longrightarrow P} \text{Init} \quad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R \quad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L \\
 \\
 \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R \quad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_R \\
 \\
 \frac{}{\Gamma \longrightarrow \top} \top R \quad \frac{}{\Gamma, \perp \longrightarrow C} \perp L
 \end{array}$$

We reformulate the system using left propositions L , right propositions R , and two new sequents $\Delta; \Omega \Longrightarrow A; \cdot$ and $\Delta; \Omega \Longrightarrow \cdot; R$. In order to prove $\Gamma \longrightarrow A$, we try to prove $\cdot; \Gamma \Longrightarrow A; \cdot$ instead. Note that L and R are not invertible.

$$\begin{array}{l}
 \text{left proposition } L ::= P \mid A \supset A \\
 \text{right proposition } R ::= P \mid A \vee A \mid \perp \\
 \text{passive antecedents } \Delta ::= \cdot \mid \Delta, L \\
 \text{active antecedents } \Omega ::= \cdot \mid \Omega, A \quad (\text{ordered sequence})
 \end{array}$$

Question 1. [15 pts] Write four inference rules for analyzing A in $\Delta; \Omega \Longrightarrow A; \cdot$:

Question 2. [20 pts] Write five inference rules for analyzing A in $\Delta; \Omega, A \Longrightarrow \cdot; R$:

Question 3. [10 pts] Write two inference rules for making a choice on R in $\Delta; \cdot \Longrightarrow \cdot; R$:

Question 4. [5 pts] Write an inference rule for making a choice on L in $\Delta, L; \cdot \Longrightarrow \cdot; R$:

4 Forward proof search [50 pts]

Consider the following fragment of the sequent calculus for propositional logic with \wedge and \supset :

$$\frac{}{\Gamma, A \longrightarrow A} \textit{Init} \quad \frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L_L \quad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \wedge L_R \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R$$

We use $\Gamma \Longrightarrow A$ in the forward sequent calculus, where we interpret Γ as a set. For example, Γ, A is equal to Γ if A is already in Γ . We also write $\Gamma \cup \Gamma'$ for the union of Γ and Γ' .

The two invariants to maintain are:

- Soundness: If $\Gamma \Longrightarrow C$, then $\Gamma \longrightarrow C$.
- Completeness: If $\Gamma \longrightarrow C$, then $\Gamma' \Longrightarrow C$ for some $\Gamma' \subset \Gamma$.

Write inference rules for the forward sequent calculus. You need a total of 7 inference rules.

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Work sheet