Name: Hemos ID:

CSE-433 Logic in Computer Science 2007
Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 175 points.
- There are 16 pages in this exam, including 3 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

	Problem 1	Problem 2	Problem 3	Problem 4	Total
Score					
Max	30	55	50	40	175

1 Admissibility of Harrop's rule [30 pts]

In constructive logic, there are some admissible rules that are not derivable. In the midterm exam, we considered Harrop's rule as an example of such a rule and showed that it is not derivable. As we said before, in this problem, we will show that Harrop's rule is indeed admissible. As a reminder, Harrop's rule is as follows:

$$\frac{\neg A \supset (B \vee C)\ \mathit{true}}{(\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}}\ \mathit{Harrop}$$

where A, B, and C are all propositions.

You will need to use the following inference rules for neutral and normal judgments.

$$\frac{\begin{array}{c}\overline{A{\downarrow}}^{\,x} \\ \vdots \\ B{\uparrow}\end{array}}{A \supset B{\uparrow}} \supset\mathsf{I}^{x}_{\uparrow} \qquad \frac{A \supset B{\downarrow} \quad A{\uparrow}}{B{\downarrow}} \supset\mathsf{E}_{\downarrow} \qquad \frac{A{\uparrow} \quad B{\uparrow}}{A \wedge B{\uparrow}} \wedge\mathsf{I}_{\uparrow} \qquad \frac{A \wedge B{\downarrow}}{A{\downarrow}} \wedge\mathsf{E}_{\mathsf{L}\downarrow} \qquad \frac{A \wedge B{\downarrow}}{B{\downarrow}} \wedge\mathsf{E}_{\mathsf{R}\downarrow}$$

$$\frac{A{\uparrow}}{A \vee B{\uparrow}} \vee\mathsf{I}_{\mathsf{L}\uparrow} \qquad \frac{B{\uparrow}}{A \vee B{\uparrow}} \vee\mathsf{I}_{\mathsf{R}\uparrow} \qquad \frac{A \vee B{\downarrow} \quad \begin{array}{c}\overline{A{\downarrow}}^{\,x} \\ \vdots \\ C{\uparrow}\end{array} \quad \begin{array}{c}\overline{B{\downarrow}}^{\,y} \\ \vdots \\ C{\uparrow}\end{array}}{C{\uparrow}} \vee\mathsf{E}^{x,y}_{\uparrow}$$

$$\frac{}{\top{\uparrow}} \top\mathsf{I}_{\uparrow} \qquad \frac{\bot{\downarrow}}{C{\uparrow}} \bot\mathsf{E}_{\uparrow} \qquad \frac{A{\downarrow}}{A{\uparrow}} {\downarrow}{\uparrow} \qquad \frac{\begin{array}{c}\overline{A{\downarrow}}^{\,x} \\ \vdots \\ \bot{\uparrow}\end{array}}{\neg A{\uparrow}} \neg\mathsf{I}_{\uparrow}{}^{x} \qquad \frac{\neg A{\downarrow} \quad A{\uparrow}}{\bot{\downarrow}} \neg\mathsf{E}_{\downarrow}$$

- You should annotate every part of your derivation with the name of the inference rule (*e.g.*, $\supset\mathsf{E}_{\downarrow}$) and also a label if applicable (*e.g.*, $\supset\mathsf{I}^{x}_{\uparrow}$). In particular, you should annotate each hypothesis with some variable (*e.g.*, $\overline{A{\downarrow}}^{\,x}$).

Since we cannot directly show that Harrop's rule is admissible, we need to analyze the proof structure of the premise, $\neg A \supset (B \vee C)\ \mathit{true}$, to show that $(\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}$ is provable. Recall that in constructive logic, $A\ \mathit{true}$ is provable if and only if $A{\uparrow}$ is provable. Therefore we will show that Harrop's rule is admissible by showing that $(\neg A \supset B) \vee (\neg A \supset C){\uparrow}$ is provable in the presence of a proof of $\neg A \supset (B \vee C){\uparrow}$.

Proof. The proof of $\neg A \supset (B \vee C){\uparrow}$, if any, must be of the following form:

$$\frac{\mathcal{D}\left\{\begin{array}{c}\overline{\neg A{\downarrow}}^{\,x} \\ \vdots \\ B \vee C{\uparrow}\end{array}\right.}{\neg A \supset (B \vee C){\uparrow}} \supset\mathsf{I}^{x}_{\uparrow}$$

We need to show that $(\neg A \supset B) \vee (\neg A \supset C){\uparrow}$ is provable by exploring *all* possible cases of the structure of $\mathcal{D}$.

Case 1. Suppose that in $\mathcal{D}$, the rule $\neg\mathsf{E}_{\downarrow}$ is applied to $\overline{\neg A\downarrow}^{\,x}$. Then there must be a proof of $A\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{\,x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

Case 2. Suppose that in $\mathcal{D}$, the rule $\vee\mathsf{I}_{\mathsf{L}\uparrow}$ is applied to deduce $B \vee C\uparrow$. Then whtere must be a proof of $B\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{\,x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

Case 3. Suppose that in $\mathcal{D}$, the rule $\vee I_{R\uparrow}$ is applied to deduce $B \vee C\uparrow$. Then there must be a proof of $C\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

The above three cases are the only possible cases of the structure of $\mathcal{D}$. Therefore in the presence of a proof of $\neg A \supset (B \vee C)\uparrow$, we can always prove $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$, which means that Harrop's rule is admissible.

2 Datatypes [55 pts]

We use the following definition of proof terms for the predicate $m < n$:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{ltI_0} \mid \mathsf{ltI_s}(M) \mid \mathsf{ltE_0}(M) \mid \mathsf{ltE_s}(M)$$

$$\frac{}{\mathsf{ltI_0} : \mathbf{0} < \mathbf{s}(n)} <\mathsf{I_0} \qquad \frac{M : m < n}{\mathsf{ltI_s}(M) : \mathbf{s}(m) < \mathbf{s}(n)} <\mathsf{I_s} \qquad \frac{M : m < \mathbf{0}}{\mathsf{ltE_0}(M) : C} <\mathsf{E_0} \qquad \frac{M : \mathbf{s}(m) < \mathbf{s}(n)}{\mathsf{ltE_s}(M) : m < n} <\mathsf{E_s}$$

We use the following definition of proof terms for the predicate $m =_{\mathsf{N}} n$:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{eqI_0} \mid \mathsf{eqI_s}(M) \mid \mathsf{eqE_{0s}}(M) \mid \mathsf{eqE_{s0}}(M) \mid \mathsf{eqE_s}(M)$$

$$\frac{}{\mathsf{eqI_0} : \mathbf{0} =_{\mathsf{N}} \mathbf{0}} =_{\mathsf{N}}\mathsf{I_0} \qquad \frac{M : m =_{\mathsf{N}} n}{\mathsf{eqI_s}(M) : \mathbf{s}(m) =_{\mathsf{N}} \mathbf{s}(n)} =_{\mathsf{N}}\mathsf{I_s}$$

$$\frac{M : \mathbf{0} =_{\mathsf{N}} \mathbf{s}(n)}{\mathsf{eqE_{0s}}(M) : C} =_{\mathsf{N}}\mathsf{E_{0s}} \qquad \frac{M : \mathbf{s}(m) =_{\mathsf{N}} \mathbf{0}}{\mathsf{eqE_{s0}}(M) : C} =_{\mathsf{N}}\mathsf{E_{s0}} \qquad \frac{M : \mathbf{s}(m) =_{\mathsf{N}} \mathbf{s}(n)}{\mathsf{eqE_s}(M) : m =_{\mathsf{N}} n} =_{\mathsf{N}}\mathsf{E_s}$$

The following proof term represents a proof by induction on natural numbers:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{ind}\ u(t)\ \mathsf{of}\ u(\mathbf{0}) \Rightarrow M \mid u(\mathbf{s}(x)) \Rightarrow N$$

Question 1. [5 pts] Define a function *append* concatenating two lists:

$$\begin{array}{rcl} \mathit{append}\ \mathbf{nil}^{\tau}\ t & = & t \\ \mathit{append}\ (x :: l)\ t & = & x :: (\mathit{append}\ l\ t) \end{array}$$

$\mathit{append} \in \mathsf{list}\ \tau \rightarrow \mathsf{list}\ \tau \rightarrow \mathsf{list}\ \tau$

$\mathit{append} \ =$

Question 2. [5 pts] Give a proof term of type $(\exists x \in \tau. A(x) \vee B(x)) \supset ((\exists x \in \tau. A(x)) \vee (\exists x \in \tau. B(x)))$. You may omit type annotations in injection terms. For example, you may write $\mathsf{inl}\ M$ when $\mathsf{inl}_A\ M$ is expected.

Question 3. [5 pts] The specification of a proof term *pred* of type $\forall x \in \mathsf{nat}. \neg(x =_{\mathsf{N}} \mathbf{0}) \supset \exists y \in \mathsf{nat}. \mathbf{s}(y) =_{\mathsf{N}} x$ is:

$$\begin{array}{lllcl} & x & v : \neg(x =_{\mathsf{N}} \mathbf{0}) & & \text{proof term of type } \exists y \in \mathsf{nat}. \mathbf{s}(y) =_{\mathsf{N}} x \\ pred & \mathbf{0} & v : \neg(\mathbf{0} =_{\mathsf{N}} \mathbf{0}) & = & \mathsf{abort}_{\exists y \in \mathsf{nat}. \mathbf{s}(y) =_{\mathsf{N}} \mathbf{0}} \ (v \ \mathsf{eqI}_0) \\ pred & \mathbf{s}(x') & v : \neg(\mathbf{s}(x') =_{\mathsf{N}} \mathbf{0}) & = & \langle x', \mathsf{eqI_s}(eqNat \ x') \rangle \end{array}$$

Here $eqNat$ is defined as $\lambda x \in \mathsf{nat}. \, \mathsf{ind} \ u(x) \ \mathsf{of} \ u(\mathbf{0}) \Rightarrow \mathsf{eqI}_0 \mid u(\mathbf{s}(x')) \Rightarrow \mathsf{eqI_s}(u(x'))$. Give a definition of *pred* based on this specification.

$pred \ =$

Question 4. [15 pts] We wish to find a proof term *ltSucc* of type

$$\forall m \in \mathsf{nat}. \forall n \in \mathsf{nat}. m < \mathbf{s}(n) \supset (m =_{\mathsf{N}} n \vee m < n)$$

(which you used in Assignment 8). Write a specification of *ltSucc* (similarly to *pred*) and then translate it to a definition as in Question 3.

Specification:

Definition:

Question 5. [5 pts] Write the elimination rule for the predicate $m =_{\mathsf{N}} n$ based on induction on predicates (which is given in the Course Notes):

$$\frac{}{A(m_0, n_0)\ \mathit{true}} =_{\mathsf{N}}\mathsf{E}_I^{w,u(m,n)}$$

Question 6. [5 pts] Use this elimination rule to prove $\forall x \in \mathsf{nat}.\forall y \in \mathsf{nat}.x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ \mathit{true}$:

$$\frac{}{\forall x \in \mathsf{nat}.\forall y \in \mathsf{nat}.x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ \mathit{true}}$$

Question 7. [5 pts] Assuming definitional equality, give a proof term of type $\forall x \in \mathsf{nat}. \forall y \in \mathsf{nat}. x + \mathbf{s}(y) =_{\mathsf{N}} \mathbf{s}(x + y)$:

Question 8. [5 pts] The following inference rule is derivable. True or false?

$$\frac{A(t)\ \mathit{true} \quad t =_{\mathsf{N}} s}{A(s)\ \mathit{true}}\ \mathit{NatEq}\mathsf{E}$$

Question 9. [5 pts] The rule $\mathit{NatEq}\mathsf{E}$ in the previous question is admissible. True or false?

3 Classical logic [50 pts]

In this problem, we use contexts Γ and Δ defined as follows:

$$\begin{array}{lcl} \Gamma & ::= & \cdot \mid \Gamma, x : A \\ \Delta & ::= & \cdot \mid \Delta, x : A\ \mathit{false} \end{array}$$

We use $\Gamma; \Delta \vdash_{\mathsf{K}} C\ \mathit{true}$ for typechecking proof terms in classical logic and $\Gamma \vdash_{\mathsf{I}} M : C$ for typechecking proof terms in constructive logic.

Here are the rules Contra↑ and Contra↓ given in the Course Notes:

$$\frac{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} A\ \mathit{true}}{\Gamma; \Delta \vdash_{\mathsf{K}} A\ \mathit{true}}\ \mathsf{Contra}\!\uparrow \qquad \frac{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} A\ \mathit{true}}{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} C\ \mathit{true}}\ \mathsf{Contra}\!\downarrow$$

Question 1. [5 pts] Use the rules Contra↑ and Contra↓ to show that the rule Peirce is derivable.

$$\frac{}{\cdot; \cdot \vdash_{\mathsf{K}} ((A \supset B) \supset A) \supset A\ \mathit{true}}$$

Question 2. [20 pts] We use the following double-negation translation for the fragment of propositional logic with implication:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation M° of a given proof term M. Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A \textit{ false} \in \Delta\}. \end{array}$$

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}$ Hyp

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\, M : A \supset B}$ $\supset$I

$(\lambda x{:}A.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}$ $\supset$E

$(M\ N)^\circ =$

Case $\dfrac{\Gamma; \Delta, x : A \textit{ false} \vdash_{\mathsf{K}} M : A}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{callcc}\ x : A \textit{ false}.\, M : A}$ Callcc

$(\mathsf{callcc}\ x : A \textit{ false}.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \quad x : A \textit{ false} \in \Delta}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{throw}\ M\ \mathsf{to}\ x : C}$ Throw

$(\mathsf{throw}\ M\ \mathsf{to}\ x)^\circ =$

Question 3. [25 pts] The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M\ N$ specifies that we finish evaluating N before we apply the function from M to the result of evaluating N.

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both A° and B°:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & \neg\neg A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation M° of a given proof term M. Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If* $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, *there exists a proof term* M° *such that* $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ *where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A\ \mathit{false} \in \Delta\}. \end{array}$$

Note the change in the definition of Γ° which now assigns to x type $\neg\neg A^\circ$. The translation of $\mathsf{callcc}\ x : A\ \mathit{false}.\ M$ and $\mathsf{throw}\ M\ \mathsf{to}\ x$ is the same as in the previous CPS translation and is omitted.

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}\ \mathsf{Hyp}$

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\ M : A \supset B}\ {\supset}\mathsf{I}$

$(\lambda x{:}A.\ M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}\ {\supset}\mathsf{E}$

$(M\ N)^\circ =$

4 Linear logic [40 pts]

For each proposition A in linear logic below, prove its truth using natural deduction with the linear hypothetical judgment $\Delta \Vdash A$. You should annotate every part of your derivation with the name of the inference rule. If its truth is unprovable, state so.

Question 1. [5 pts] $(A \otimes \mathbf{1}) \multimap A$

Question 2. [5 pts] $(A \oplus \mathbf{0}) \multimap A$

Question 3. [5 pts] $(A \mathbin{\&} A) \multimap A$

Question 4. [5 pts] $(A \oplus A) \multimap A$

Question 5. [5 pts] $(A \oplus (B \otimes C)) \multimap ((A \oplus B) \otimes (A \oplus C))$ *true* is provable. True or false?

Question 6. [5 pts] $(A \oplus (B \,\&\, C)) \multimap ((A \oplus B) \,\&\, (A \oplus C))$ *true* is provable. True or false?

Question 7. [5 pts] $(!A \otimes !B) \multimap (A \,\&\, B)$ *true* is provable. True or false?

Question 8. [5 pts] $((A \multimap !A) \,\&\, (B \multimap !B)) \multimap ((A \otimes B) \multimap !(A \otimes B))$ *true* is provable. True or false?

Work sheet

Work sheet

Work sheet