# CSE-433 Assignment - *Substitution*

gla@postech

Due at class, Thursday, Sep 29

- For this assignment, do not discuss proof ideas and techniques with your classmates.
- Please write your proofs clearly and legibly.

Consider the natural deduction system using hypothetical judgments:

$$\frac{A\ true \in \Gamma}{\Gamma \vdash A\ true}\ \mathsf{Hyp} \qquad \frac{\Gamma, A\ true \vdash B\ true}{\Gamma \vdash A \supset B\ true}\ {\supset}\mathsf{I} \qquad \frac{\Gamma \vdash A \supset B\ true \quad \Gamma \vdash A\ true}{\Gamma \vdash B\ true}\ {\supset}\mathsf{E}$$

$$\frac{\Gamma \vdash A\ true \quad \Gamma \vdash B\ true}{\Gamma \vdash A \wedge B\ true}\ {\wedge}\mathsf{I} \qquad \frac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash A\ true}\ {\wedge}\mathsf{E_L} \qquad \frac{\Gamma \vdash A \wedge B\ true}{\Gamma \vdash B\ true}\ {\wedge}\mathsf{E_R}$$

$$\frac{\Gamma \vdash A\ true}{\Gamma \vdash A \vee B\ true}\ {\vee}\mathsf{I_L} \qquad \frac{\Gamma \vdash B\ true}{\Gamma \vdash A \vee B\ true}\ {\vee}\mathsf{I_R} \qquad \frac{\Gamma \vdash A \vee B\ true \quad \Gamma, A\ true \vdash C\ true \quad \Gamma, B\ true \vdash C\ true}{\Gamma \vdash C\ true}\ {\vee}\mathsf{E}$$

$$\frac{}{\Gamma \vdash \top\ true}\ {\top}\mathsf{I} \qquad \frac{\Gamma \vdash \bot\ true}{\Gamma \vdash C\ true}\ {\bot}\mathsf{E}$$

The proof of the substitution theorem goes as follows:

**Proposition 0.1 (Structural properties).**
(Weakening) *If* $\Gamma \vdash C\ true$, *then* $\Gamma, A\ true \vdash C\ true$.
(Contraction) *If* $\Gamma, A\ true, A\ true \vdash C\ true$, *then* $\Gamma, A\ true \vdash C\ true$.

**Theorem 0.2 (Substitution).** *If* $\Gamma \vdash A\ true$ *and* $\Gamma, A\ true \vdash C\ true$, *then* $\Gamma \vdash C\ true$.

*Proof.* By induction on the structure of the proof of $\Gamma, A\ true \vdash C\ true$.

Case $\dfrac{C\ true \in \Gamma}{\Gamma, A\ true \vdash C\ true}\ \mathsf{Hyp}$

$\Gamma \vdash C\ true$ — by the rule $\mathsf{Hyp}$ with $C\ true \in \Gamma$

Case $\dfrac{}{\Gamma, A\ true \vdash C\ true}\ \mathsf{Hyp}$ where $A = C$

$\Gamma \vdash C\ true$ — from the assumption $\Gamma \vdash A\ true$

Case $\dfrac{\Gamma, A\ true, C_1\ true \vdash C_2\ true}{\Gamma, A\ true \vdash C_1 \supset C_2\ true}\ {\supset}\mathsf{I}$ where $C = C_1 \supset C_2$

$\Gamma, C_1\ true \vdash A\ true$ — by weakening $\Gamma \vdash A\ true$
$\Gamma, C_1\ true \vdash C_2\ true$ — by IH on $\Gamma, A\ true, C_1\ true \vdash C_2\ true$ with $\Gamma, C_1\ true \vdash A\ true$
$\Gamma \vdash C_1 \supset C_2\ true$ — by the rule ${\supset}\mathsf{I}$ with $\Gamma, C_1\ true \vdash C_2\ true$

Case $\dfrac{\Gamma, A\ true \vdash C' \supset C\ true \quad \Gamma, A\ true \vdash C'\ true}{\Gamma, A\ true \vdash C\ true}\ {\supset}\mathsf{E}$

$\Gamma \vdash C' \supset C\ true$ — by IH on $\Gamma, A\ true \vdash C' \supset C\ true$ with $\Gamma \vdash A\ true$
$\Gamma \vdash C'\ true$ — by IH on $\Gamma, A\ true \vdash C'\ true$ with $\Gamma \vdash A\ true$
$\Gamma \vdash C\ true$ — by the rule ${\supset}\mathsf{E}$ with $\Gamma \vdash C' \supset C\ true$ and $\Gamma \vdash C'\ true$

⋮

□

Give the cases for the rules ${\vee}\mathsf{I_L}$, ${\vee}\mathsf{E}$, and ${\bot}\mathsf{E}$. You may use the two structural properties without proofs.